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LETTER TO THE EDITOR

Series approach to the bond bending model

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Abstract. The bond bending model is studied using the series expansion method on a honeycomb lattice. The elastic splay susceptibility χ_{SR} and the elastic compressional susceptibility χ_{el} are calculated up to 18th order. The elastic splay crossover exponent, ζ_{SP}, is found to be $\zeta_{SP} = 1.31 \pm 0.02$ which is very close to the conductivity exponent, ζ_{Re}, of the resistor network. From the scaling relation $f_B = d\nu + \zeta_{SP}$, we found that the bulk modulus exponent $f_B = 3.98 \pm 0.02$ which is in excellent agreement with the result $f_B = 3.96 \pm 0.04$, obtained by Zabolitzky *et al* using a transfer matrix technique on the same lattice.

Recently much attention [1–6] has been directed towards randomly diluted elastic network. Various models have been studied, such as the central force model [1, 3, 5], the bond bending model [2, 4], and the granular disk model [7, 8], etc. The bond bending model is perhaps the best understood. It has been suggested [4, 9, 10, 11] that, for the bond-bending model, one has the relation

$$f_B = t + 2\nu \tag{1}$$

where ν is the correlation length exponent for percolation and $t = (d-2)\nu + \zeta_{Re}$ [12] is the exponent for the conductivity of the analogous randomly diluted resistor network defined by $\Sigma(p) \sim \sigma|p - p_c|^t$, where σ is the conductance of an occupied bond (which occurs with probability p) and the vacant bonds occurring with probability $1-p$ have zero conductance. Here ζ_{Re} is the conductivity crossover exponent for the resistor network. In terms of this exponent equation (1) can be written as $f_B = d\nu + \zeta_{Re}$. Zabolitzky *et al* [13] have calculated f_B for the bond bending model on a honeycomb lattice by computer simulation using a transfer matrix method. They found that $f_B = 3.96 \pm 0.04$. Using $t = 1.30$ [14] (or equivalently $\zeta_{Re} = 1.30$) and $\nu = \frac{4}{3}$, we see that equation (1) is almost exact. Recently, there have been some efforts to understand the physics behind (1). For instance, it has been proved [15] that the elastic splay crossover exponent [5] ζ_{SP} of the bond bending model, which is related to the bulk modulus exponent $f_B = d\nu + \zeta_{SP}$ [11, 8, 15], is the same as the conductivity crossover exponent ζ_{Re} for the lattice animal. Using the series expansion method the elastic splay crossover exponent ζ_{SP} has been calculated [16] on a honeycomb lattice up to 13th order. Due to the shortness of the series, the crossover exponent ζ_{SP} could not be accurately determined. In this letter, we [17] have extended the series to 18th order, which enables us to estimate the exponent very accurately. We found that $\zeta_{SP} = 1.31 \pm 0.02$, which is in excellent agreement with ζ_{Re}. Our numerical result indicates that the elastic splay crossover exponent ζ_{SP} and conductivity crossover exponent ζ_{Re} are the same in two dimensions.

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Consider the randomly diluted bond bending model in two dimensions whose Hamiltonian can be written as

$$H = \tfrac{1}{2}k_{\rm cf}\sum_{b}|u_b\cdot\hat{R}_b|^2\epsilon_b + \tfrac{1}{2}k_{\rm bb}\sum_{\langle b,b'\rangle}|u_b\times\hat{R}_b - u_{b'}\times\hat{R}_{b'}|^2\epsilon_b\epsilon_{b'} \tag{2}$$

where b labels bond and $\langle b,b'\rangle$ denotes that the sum is over pairs of nearest-neighbour bonds. Also $u_b = u_s - u_{s'}$, $u_{b'} = u_s - u_{s''}$, where the s label sites at the ends of the bonds, u_s is the displacement at site s, $k_{\rm cf}$ is the central force elastic constant, $k_{\rm bb}$ is the bond-bending elastic constant, $\hat{R}_b$ is a unit vector along the nearest-neighbour direction, and ϵ_b is an indicator variable which is one with probability p and zero with probability $1-p$. Note that the first term in (2) is the Hamiltonian for the central force model. The second term of (2) can be written as

$$H_{\rm angle} = \tfrac{1}{2}k_{\rm bb}\sum_{\langle b,b'\rangle}(\theta_b-\theta_{b'})^2\epsilon_b\epsilon_{b'} \tag{3}$$

where $\theta_b = (u_b\times\hat{R}_b)\cdot\hat{k}$ is the angular displacement of the bond b and $\hat{k}$ is a unit vector perpendicular to the plane of the lattice.

In a manner similar to the two-point resistive susceptibility $\chi_{\rm Re}$ introduced by Harris and Fisch [12] in calculating the conductivity crossover exponent $\zeta_{\rm Re}$ in a resistor network, we [5, 16] define the two-bond splay susceptibility $\chi_{\rm SR}(b,b')$ for the elastic network as

$$\chi_{\rm SR}(b,b') = [Tr\,\Psi_\lambda(b)\Psi_{-\lambda}(b')e^{-H}/Tre^{-H}]_{\rm av} \tag{4}$$

where $\Psi_\lambda(b) = \exp(i\lambda(u_b\times\hat{R}_b)\cdot\hat{k}) = \exp(i\lambda\theta_b)$ is the splay order parameter and $[\cdots]_{\rm av}$ denotes the average over the random variables ϵ_b, Tr indicates integration over all displacements. Denoting by $\chi^{\rm SR}_{b,b'}$ the effective bond angle elastic resistance (i.e. the inverse of the elastic constant) for the splay distortion defined as the angular displacement of the two bonds divided by the torque, we have [5]

$$\chi^{\rm SR}_{b,b'} = (\theta_b-\theta_{b'}|G|\theta_b-\theta_{b'}) \tag{5}$$

where G is the Green function of the bond bending model defined as $G = \lim_{\epsilon\to\infty}(V+i\epsilon)^{-1}$, in which V is the dynamical matrix. From equations (4) and (5), we obtain

$$\chi_{\rm SR}(b,b') = \left[\exp\left(-\frac{\lambda^2}{2k}\tilde{\chi}^{\rm SR}_{b,b'}\right)\right]_{\rm av} \tag{6}$$

where $\tilde{\chi}^{\rm SR}_{b,b'} = k\chi^{\rm SR}_{b,b'}$ and $k\sim k_{\rm cf}\sim k_{\rm bb}$. Note that $\chi^{\rm SR}_{b,b'} = \infty$ if bonds b and b' are not in the same cluster, so that

$$\chi_{\rm SR}(b,b') = \left[\nu_{b,b'}\exp\left(-\frac{\lambda^2}{2k}\tilde{\chi}^{\rm SR}_{b,b'}\right)\right]_{\rm av} \tag{7}$$

where $\nu_{b,b'}$ is the pair-connectedness function. For small λ^2/k, we expand equation (7) in powers of λ^2/k:

$$\chi_{\rm SR}(b,b') = \left[\nu_{b,b'}\left(1-\frac{\lambda^2}{2k}\tilde{\chi}^{\rm SR}_{b,b'}+\cdots\right)\right]_{\rm av} \sim \chi_p(b,b')\left(1-\frac{\lambda^2}{2k}|r_b-r_{b'}|^{\zeta_{\rm SP}/\nu}\right) \tag{8}$$

where $\zeta_{\rm SP}$ is the elastic splay crossover exponent which describes the way $[\chi^{\rm SR}_{b,b'}]_{\rm av}$ scales with the distance, i.e. $[\chi^{\rm SR}_{b,b'}]_{\rm av}\sim|r_b-r_{b'}|^{\zeta_{\rm SP}/\nu}$ and $\chi_p(b,b')$ is the susceptibility for percolation. In the limit $k\to\infty$, $\chi_{\rm SR}(b,b')$ defines the percolation problem. Therefore, the elastic problem of the bond bending model can be described by this crossover exponent $\zeta_{\rm SP}$. The crossover exponent $\zeta_{\rm el}$ describing the way two-point elastic constant scales with the distance can be defined in a similar way [15, 16].

Using a scaling argument [11, 15, 16] the exponents ζ_{SP} and ζ_{el} can be related to the elastic bulk modulus exponent f_B:

$$f_B = d\nu + \zeta_{SP} \tag{9}$$

$$f_B = (d-2)\nu + \zeta_{el}. \tag{10}$$

From which we have

$$\zeta_{el} = 2\nu + \zeta_{SP}. \tag{11}$$

To calculate the exponent ζ_{SP} and ζ_{el}, we have developed series expansions in powers of p up to p^{18} on the honeycomb lattice for the following quantities [5, 16]:

$$\chi_{SR} = \left[\sum_b \chi^{SR}_{b,b'}\right]_{av} \equiv \sum_\Gamma P(\Gamma) \sum_{b\in\Gamma} \chi^{SR}_{b,b'}$$

$$\equiv \sum_n a_n p^n \sim |p - p_c|^{-\gamma-\zeta_{SP}} \tag{12}$$

$$\chi_{el} = \left[\sum_x \chi^{el}_{x,x'}\right]_{av} \equiv \sum_\Gamma P(\Gamma) \sum_{x\in\Gamma} \chi^{el}_{x,x'}$$

$$\equiv \sum_n b_n p^n \sim |p - p_c|^{-\gamma-\zeta_{el}} \tag{13}$$

where the summation $\sum_\Gamma$ is over all clusters, $P(\Gamma)$ is the associated probability per site that the cluster Γ occurs. For the resistor network, two clusters which are topologically equivalent give the same two-point resistance. For the elastic network, however, clusters with different shapes give different values of elastic response. In calculating the series for χ_{SR} and χ_{el} we have to count every cluster with different shape, which is very time consuming.

The series up to order p^{18} on the honeycomb lattice are presented in table 1. For the bond bending model the rigidity percolation threshold is the same as that for the usual bond percolation on the honeycomb lattice, namely $p_c = 0.6527$. So it is much easier to analyse the series.

We have analysed the series with two different methods [21], one based on the assumption that there are non-analytic confluent corrections to scaling and another [22] based on the assumption that there are logarithmic confluent corrections. Non-analytic confluent corrections to scaling [13] have several origins, including irrelevant operators. They are definitely present in both isotropic and directed two-dimensional percolation and thus must be allowed for in the series discussed in this letter. There is also evidence from simulation by Zabolitzky *et al* [13] that there may be logarithmic corrections to elastic critical behaviour in two dimensions.

To analyse the series, denoted in general by $\chi(p)$, we assume that the series has the form

$$\chi(p) \sim a(p_c - p)^{-h}[1 + b(p_c - p)^{\Delta_1} + \cdots] \qquad p < p_c \tag{14}$$

where h is the critical exponent that we wish to determine, and p_c is the critical point. We first transform the series in p into the series in the variable y, where

$$y = 1 - (1 - p/p_c)^{\Delta_1} \tag{15}$$

and then take Padé approximants [23] to

$$G(y) = \Delta_1(y-1)\frac{d}{dy}\ln(\chi) \tag{16}$$

Table 1. Series coefficients a_n and b_n.

n	a_n	b_n
1	3.0000	0.0000
2	10.5000	6.0000
3	32.9286	22.0000
4	100.0714	64.0000
5	290.4774	168.0000
6	602.9276	337.5000
7	1576.3939	785.2813
8	3805.2511	1661.4965
9	8916.5410	3434.4097
10	16 571.0220	5714.8813
11	42 337.3310	13 308.6716
12	82 386.6181	22 706.8160
13	175 877.6996	43 003.3083
14	334 729.4822	73 305.3381
15	763 065.2709	157 338.7425
16	1 301 615.2032	224 620.4520
17	2 916 663.1815	485 147.8259
18	4 984 470.7163	717 171.4318

which should converge to $-h$. Since we already know the critical point p_c we can simply plot graphs of h versus the input Δ_1 and choose Δ_1 such that all Padé approximants give as closely as possible the same values of h.

The assumption of logarithmic corrections entails fitting to the form

$$\chi(p) \sim (p_c - p)^{-h} |\ln(p_c - p)|^{\theta} \qquad p < p_c. \tag{17}$$

We fitted this form with the method of Adler and Privman [22]. The analysis of the logarithmic form involves taking Padé approximants to the series

$$H(p) = -(p_c - p)|\ln(p_c - p)|\{(\chi'/\chi) - [h/(p_c - p)]\} \tag{18}$$

so that when $H(p)$ goes to θ as $p \to p_c$. To get the exponent h, we take Padé approximants to $H(p)$ at p_c to obtain graphs of θ as a function of h.

From our previous experience [16], the convergence of the series χ_{SR} is better than χ_{el}. We have calculated the Padé approximants from the χ''_{SR} series assuming the scaling form given in (14), where χ''_{SR} is the second derivative of the series χ_{SR}. All the Padé approximants converge around $\Delta_1 = 3.0$, from which we estimate $h = \gamma + \zeta_{SP} + 2 = 5.70 \pm 0.02$ or $\zeta_{SP} = 1.31 \pm 0.02$, where we have used the exact value $\gamma = 2.3888\ldots$ [24] and the error bar is somewhat subjective. We have also tested the logarithmic corrections for the series χ_{SR} and found no convergence. For the series χ_{el} we have fitted the series χ'_{el} to the form of (14) and found that $h = \gamma + \zeta_{el} + 1 \sim 6.17$ or $\zeta_{el} \sim 2.78$ which is clearly wrong. If we enforce the logarithmic corrections, however, we find that $h = \gamma + \zeta_{el} + 1 = 7.39 \pm 0.05$ and $\theta = -1.0$. This gives $\zeta_{el} = 4.00 \pm 0.05$ which satisfies relation (11). Since there is an indication [13] that the logarithmic corrections exist in the bond bending model, we favour the result $\zeta_{el} = 4.00 \pm 0.05$. The reason that the series χ_{el} gives a poor result is the following: in calculating the series χ_{el}, we calculate the effective two-point elastic resistance $\chi^{el}_{x,x'} \sim u_{x,x'}/F_{x,x'}$ between sites x and x', where $u_{x,x'}$ is the displacement along $\hat{r}_{x,x'}$, the unit vector connecting sites x and x' and $F_{x,x'}$ is the force along $\hat{r}_{x,x'}$ needed to maintain the equilibrium. Hence $\chi^{el}_{x,x'}$ depends on $\hat{r}_{x,x'}$ which will affect the result for small systems or small clusters. Using $\zeta_{SP} = 1.31 \pm 0.02$ and equation (9), we obtain $f_B = 3.98 \pm 0.02$, which

is in excellent agreement with equation (1) and with the exponent calculated by Zabolitzky *et al.*

In summary, we have calculated the elastic crossover exponents, ζ_{SP} and ζ_{el}, for the bond bending model using the series expansion method on the honeycomb lattice. We found that $\zeta_{SP} = 1.31 \pm 0.02$ and $\zeta_{el} = 4.00 \pm 0.05$. From the scaling theory $f_B = d\nu + \zeta_{SP}$, we obtain $f_B = 3.98 \pm 0.02$ which is in excellent agreement with equation (1) and the exponent calculated by Zabolitzky *et al.* When enforcing the logarithmic correction, the exponent ζ_{el} is consistent with the scaling theory $\zeta_{el} = 2\nu + \zeta_{SP}$.

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